CUBIC AND QUARTIC INTEGRALS FOR GEODESIC FLOW ON 2-TORUS VIA SYSTEM OF HYDRODYNAMIC TYPE

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Abstract. In this paper we deal with the classical question of existence of polynomial in momenta integrals for geodesic flows on the 2-torus. For the quasi-linear system on coefficients of the polynomial integral we consider the region (so called elliptic regions) where there are complex-conjugate eigenvalues. We show that for quartic integrals in the other two eigenvalues are real and genuinely nonlinear. This observation together with the property of the system to be Rich (Semi-Hamiltonian) enables us to classify elliptic regions completely. The case of complex-conjugate eigenvalues for the system corresponding to the integral of degree 3 is done similarly. These results show that if new integrable examples exist they could be found only within the region of Hyperbolicity of the quasi-linear system.

1. Introduction

Let ρ be a Riemannian metric on the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\Gamma$, ρ^t denotes the geodesic flow. Let $F_n : T^*\mathbb{T}^2$ be a function on the cotangent bundle which is homogeneous polynomial of degree n with respect to the fibre (notice that this condition is invariant with respect to the change of coordinates on the configuration space $\mathbb{T}^2$). We are looking for such an F_n which is an integral of motion for the geodesic flow ρ^t, i.e. $F_n \circ \rho^t = F_n$. This question leads immediately to a system of quasi-linear equations on coefficients of F_n and this is the aim of the present paper to study it for the degrees $n = 3, 4$. Let us mention that there are classically known examples of the geodesic flows on the 2-torus which have integrals F of degree one and two. These examples can be most naturally described with the use of the conformal coordinates on the covering plane (we refer to the book by Bolsinov, Fomenko [5] for the

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history and discussion of this classical question)

$$(c) \qquad ds^2 = \Lambda(q_1, q_2)(dq_1^2 + dq_2^2),\ H = \frac{1}{2\Lambda}(p_1^2 + p_2^2).$$

Here Λ is a positive function periodic with respect to the lattice Γ. With these coordinates it can be shown that the cases of linear and quadratic integral for the geodesic flow correspond to those conformal factors Λ which can be written as a sum of two functions of one variable:

$$\Lambda = f_1(m_1q_1 + n_1q_2) + f_2(m_2q_1 + n_2q_2) \quad with \quad m_1m_2/n_1n_2 = -1,$$

where one of the functions must be constant in the case of linear in momenta integral. Such metrics are called Liouville metrics. We shall call a polynomial integral F_n reducible if it can be written as a polynomial function of the Hamiltonian H and some other polynomial integral of degree smaller than n, and irreducible in the other case. Let us mention that there are no known examples of Riemannian metrics on the 2-torus having irreducible integrals of degrees higher than two. This question is related also to the so called Birkhoff conjecture on integrable convex billiards in the plane (see also [1], [16]). In [12] (see also [13]) Kozlov and Denisova proved that if Λ is trigonometric polynomial then the geodesic flow has no irreducible polynomial integrals of degree higher than two. Amazingly there do exist non-trivial examples of geodesic flows on 2-sphere with integrals which are homogeneous polynomials of degrees 3 and 4. These examples (see [14],[9],[6], [8],[18], [19]) were inspired by classical integrable cases of Goryachev-Chaplygin and Kovalevskaya in rigid body dynamics. Let us mention also that for surfaces of genus higher than one by Kozlov's theorem there are no nontrivial analytic integrals of the geodesic flow [11].

In what follows we shall work in other global coordinate system on the torus called semi-geodesic (or equidistant). It is built with the help of one regular invariant torus of the geodesic flow, call it L, which projects diffeomorphically to the configuration space $\mathbb{T}^2$. It was proved in [2] that such an invariant torus always exists and the Riemannian metric can be written in the form

$$(s) \qquad ds^2 = g^2(t, x)dt^2 + dx^2,\ H = \frac{1}{2}\left(\frac{p_1^2}{g^2} + p_2^2\right),$$

where the family $\{t = const\}$ is the family of geodesics of the chosen invariant torus. Amazingly for our approach both coordinates will play a very important role while each of (c) and (s) have their own advantages. In both coordinate systems the condition of flow-invariance can be reduced to a quasi-linear system of equations on the coefficients of polynomial F_n. However for the case (c) this system gets the form

$$A_1(U)U_{q_1} + A_2(U)U_{q_2} = 0 \tag{1}$$

while fore the case (s) it has a form of evolution equations:

$$U_t + A(U)U_x = 0. \qquad (2)$$

Here U is a vector function of coefficients and $A_i(U)$ and $A(U)$ are $n \times n$ matrices. Let us mention here the advantage of the second system. Characteristics of it are always transversal to the $x-$direction while for the second they might rotate. This fact complicates of the analysis, but this is interesting open question to find suitable generalizations for the first system as well.

It was proved in our recent paper [4] that the system (2) is in fact Rich or Semi-Hamiltonian system. Among other things this enables following P. Lax (see Serre's book [15]) to analyze blow up of smooth solutions along characteristics. Such an analysis was performed for other quasi-linear system (a reduction of Benney' chain) in [3]. Here we shall apply the same ideas to the system (2). As usual, for such type of systems one does not knows a-priori that the system is Hyperbolic or it may have regions with Complex eigenvalues. In this paper we shall concentrate on the case of degrees 3 and 4. Our main results are for these so called "elliptic" regions, we shall denote them by Ω_e, these are regions on the configuration space where the matrix $A(U)$ has all eigenvalues different and two of them are complex-conjugate. Then it follows that for $n = 3$ the third one is obviously real. Also for $n = 4$ the other two eigenvalues must be real as well (see later). Our main results for the geodesic flow says that for $n = 3, 4$ the "elliptic" regions are "standard" that is the metric on them is classically integrable:

Theorem 1. *Let $n = 3$, then one has the following alternative:*

Either metric is flat in the region Ω_e or F_3 is reducible on Ω_e , that is it can be written as combination of H and F_1

$$F_3 = k_1 F_1^3 + 2k_2 H F_1$$

for some explicit constants k_1, k_2.

Corollary 1. *We have for the conformal model (c):*

Either metric ρ is flat on Ω_e or $\Lambda = \Lambda(mq_1 + nq_2)$ on Ω_e for some reals m, n; If in addition ρ is known to be real analytic metric on $\mathbb{T}^2$ then $\Lambda = \Lambda(mq_1 + nq_2)$ everywhere on the whole torus $\mathbb{T}^2$ and the flow ρ^t necessarily has a first power integral on the whole torus $\mathbb{T}^2$.

Theorem 2. *Let $n = 4$, then the following alternative holds: Either metric ρ is flat on Ω_e or F_4 is reducible, that is it can be expressed on Ω_e as*

$$F_4 = k_1 F_2^2 + 2k_2 H F_2 + 4k_3 H^2$$

where F_2 is a polynomial of degree 2 which is an integral of the geodesic flow on Ω_e and k_i are constants.

Corollary 2. *The conformal factor $\Lambda(q_1, q_2)$ can be written on Ω_e in the form*

$$\Lambda(q_1, q_2) = f(m_1 q_1 + n_1 q_2) + g(m_2 q_1 + n_2 q_2) \quad \textit{with} \quad \frac{m_1}{n_1}\frac{m_2}{n_2} = -1.$$

If in addition Λ is known to be real analytic then Λ can be written in such a form for all q_1, q_2 on $\mathbb{T}^2$.

There are several main ingredients in the proof of these results. The first is the property of the quasi-linear system to be Rich or Semi-Hamiltonian. This means that it can be written in Riemann invariants on one hand and in the form of the conservation laws on the other hand. This was proved in our paper [4]. Next we use strong maximum principle for Riemann invariants corresponding to Complex eigenvalues. Next we were able to show that for real eigenvalues the condition of genuine nonlinearity is satisfied.

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2. Preparations

In this section we explain first the geometric meaning of the characteristic polynomial of the matrix $A(U)$ and also prove a crucial Lemma about it.

The matrix of the system (2) is the following $n \times n$ (see [4]) matrix A:

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & a_1 \\ a_{n-1} & 0 & \dots & 0 & 0 & 2a_2 - na_0 \\ 0 & a_{n-1} & \dots & 0 & 0 & 3a_3 - (n-1)a_1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{n-1} & 0 & (n-1)a_{n-1} - 3a_{n-3} \\ 0 & 0 & \dots & 0 & a_{n-1} & na_n - 2a_{n-2} \end{pmatrix} \quad (3),$$

where it is convenient to write

$$F_n = \sum_{k=0}^{n} a_k(t, x) \frac{p_1^{n-k}}{g^{n-k}} p_2^k,$$

is the integral of degree n and the unknown vector function $U = (a_0, \dots, a_{n-2}, a_{n-1})^T$ is a column vector of non-constant coefficients of F_n, where one can show that the highest two coefficients can be normalized to be

$$a_{n-1} \equiv g \quad \textit{and} \quad a_n \equiv 1.$$

The first interesting property of this system is the fact that its eigenvalues have very precise geometric meaning: it can be shown that in order to compute eigenvalues of the system one just have to find critical points of F_n restricted to the circular fibre of the energy level. In other words let us differentiate F_n along the fibre of the energy level:

Let G_n be the homogeneous polynomial which is the derivative of F_n in the direction of the fibre $\{H = 1/2\} \cap T^*_m\mathbb{T}^2$.

$$G_n = L_v(F_n),$$

where the vector field v looks differently for the coordinates (c) and (s): In case (c)

$$v = -p_2\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial p_2},$$

and in case (s)

$$v = -p_2\frac{\partial}{\partial(p_1/g)} + \frac{p_1}{g}\frac{\partial}{\partial p_2}.$$

Define now usual polynomials $\hat{G}_n$ and $\hat{F}_n$ corresponding to G_n and F_n of the variable s in both models as follows: in case (c):

$$s = \frac{p_2}{p_1}, \quad \hat{G}_n(s) = \frac{1}{p_1^n}G_n, \quad \hat{F}_n(s) = \frac{1}{p_1^n}F_n,$$

and in case (s)

$$s = \frac{p_2}{p_1/g}, \quad \hat{G}_n(s) = \frac{g^n}{p_1^n}G_n, \quad \hat{F}_n(s) = \frac{g^n}{p_1^n}F_n.$$

It turns out that with these notations eigenvalues of the matrix $A(U)$ are related to the roots s_i of $\hat{G}_n$(of the model (s)) by

$$\lambda_i = gs_i.$$

Moreover it is remarkable fact that the system (2) can be written in Riemann invariants:

$$(r_i)_t + \lambda_i(r_i)_x = 0,\ i = 1, .., n,$$

where r_i are just critical values of F_n on the circular fibres. Therefore for r_i one has the following

$$r_i = \frac{p_1^n}{g^n}\hat{F}_n(s_i) = \frac{\hat{F}_n(s_i)}{(1+s_i^2)^{n/2}}.$$

Notice that roots of $\hat{G}_n$ could be $\pm i$ then the Riemann invariant r_i would have singularity. However this does not happen by the following lemma and its corollary. Notice that both statements are invariantly formulated but to prove it we use the model (c).

Lemma 1. *Assume F_n is divisible by H for a point on $\mathbb{T}^2$. Then F_n can be represented globally as $F_n = HF_{n-2}$. Saying differently if F_n is irreducible integral then F_n is not divisible by H for any point.*

Proof. This follows in fact from two identities found by Kolokoltsov in [10] for conformal model (c). Write F_n and H in a complex way as follows

$$p = p_1 - ip_2, \ F_n = \sum A_i p^{n-i} \bar{p}^i, \ H = \frac{1}{2\Lambda} p\bar{p}.$$

It is proved in [10] that A_0 and A_n are holomorphic and thus must be constant on the torus. Therefore if F_n is divisible by H at some point on the torus then A_0, A_n must vanish at this point and hence everywhere on the torus $A_0 = A_n = 0$. The claim follows. □

Corollary 3. *For irreducible F_n, write the G_n to be derivative of F_n with respect to the fibre, then G_n is not divisible by H. In other words $\pm i$ are never among the roots of $\hat{G}_n$.*

Proof. We shall use the coordinates (c) for the proof: In complex notations $p, \bar{p}$ one has

$$G_n = \frac{i}{2}\left(\bar{p}\frac{\partial F_n}{\partial \bar{p}} - p\frac{\partial F_n}{\partial p}\right).$$

If G_n would be divisible by $p\bar{p}$ one would have $A_0 = A_n = 0$. The claim follows. □

3. Maximum principle for complex Riemann invariants

Let us recall that we consider for the case $n = 3, 4$ those regions on the configuration space where all eigenvalues are different and two of them are not real. In this case the third one is real for $n = 3$. Also for $n = 4$ the other two must be real as well. This is because any function on the circle must have maximum and minimum and eigenvalues as we explained in the previous section correspond to critical points of F_n on the fibre. Moreover the points of maximum and minimum must be different since it is impossible for F_n to be constant on a fibre. Since this would contradict the existence of the regular torus L (the point of intersection of such a fibre with L would be critical point for F_n). We shall call these regions "elliptic" regions. On the boundary $\partial\Omega_e$ the complex conjugate root collide and become real. We shall use the strong maximum principle for the following

Theorem 3. *Let $s_{1,2} = \alpha \pm i\beta$ be complex conjugate roots of the polynomial $\hat{G}_n, n = 3, 4$. Denote by*

$$r_{1,2} = u \pm iv = \frac{\hat{F}(s)}{(1 + s_{1,2}^2)^{n/2}}$$

the corresponding Riemann invariants. Then u and v must be constants on Ω_e. Moreover if Ω_e has a nontrivial boundary then $v \equiv 0$ on Ω_e.

Proof. Consider first the case $n = 4$. Then there is no square root in the denominator of $r_{1,2}$ and hence by the lemma of previous section u, v are smooth in the interior and continuous up to the boundary of Ω_e.

In the interior $r = u + iv$ satisfies the following $r_t + (g\alpha + ig\beta)r_x = 0$ therefore

$$(u + iv)_t + (g\alpha + ig\beta)(u + iv)_x = 0.$$

Then denoting $\tilde{\alpha} = g\alpha, \tilde{\beta} = g\beta$ and reducing the system

$$\begin{cases} u_t + \tilde{\alpha} u_x - \tilde{\beta} v_x = 0 \\ v_t + \tilde{\beta} u_x + \tilde{\alpha} v_x = 0 \end{cases}$$

to

$$\begin{cases} u_t - \frac{\tilde{\alpha}}{\tilde{\beta}} v_t - \frac{\tilde{\alpha}^2 + \tilde{\beta}^2}{\tilde{\beta}} v_x = 0 \\ u_x + \frac{1}{\tilde{\beta}} v_t + \frac{\tilde{\alpha}}{\tilde{\beta}} v_x = 0 \end{cases}$$

and eliminating u one arrives to the second order equation

$$\left(\frac{v_t}{\tilde{\beta}}\right)_t + \left(\frac{\tilde{\alpha}}{\tilde{\beta}} v_x\right)_t + \left(\frac{\tilde{\alpha}}{\tilde{\beta}} v_t\right)_x + \left(\frac{\tilde{\alpha}^2 + \tilde{\beta}^2}{\tilde{\beta}} v_x\right)_x = 0.$$

Its principal part has negative discriminant thus the equation is elliptic. By the strong maximum principle v cannot attain maximum in the interior point. Therefore v must be constant and moreover to be zero if there is a boundary, because on the boundary v vanishes. From the equations it follows that u must be a constant as well.

For the case $n = 3$, due to the square root in the formula one might have not a single-valued function for r. However in this case we shall consider r^2 instead, which is also a Riemann invariant and apply the same argument as above for r^2. We have that r must be a constant. □

4. Proof of the main theorem for $n = 3$

In this section we prove Theorem 1 for case n=3. The proof requires the following:

Theorem 4. *Let Ω_e be region on $\mathbb{T}^2$ with the property the polynomial*

$$\hat{G}_3(s) = a_2 s^3 + (2a_1 - 3)s^2 + (3a_0 - 2a_2)s - a_1$$

has one real and two complex conjugate roots. Then it follows that this region is a strip on the covering plane $\mathbb{R}^2(t, x)$ with the slope λ and on this strip the Riemannian metric admits 1-parametric group of symmetries $g(t, x) = g(\lambda t - x)$ and therefore there exist a linear integral $F_1 = p_1 + \lambda p_2$.

Proof. Firstly we have by Theorem 3 that r_1, r_2 are constants on Ω_e. On the domain Ω_e r_1, r_2, r_3 are coordinates in the space of field variables. r_1, r_2 being constant imply that third one satisfies the equation $(r_3)_t + \lambda_3 (r_3)_x = 0$, where $\lambda_3 = s_3 g$. Since r_1, r_2 are constant on Ω_e then λ_3 depends only on r_3. We have got the simplest quasi-linear equation

$$(r_3)_t + \lambda_3(r_3)(r_3)_x = 0.$$

Now the characteristic of this equations are integral curves of the vector field

$$\frac{\partial}{\partial t} + \lambda_3(r_3(t,x))\frac{\partial}{\partial x}$$

and r_3 must be constant on these curves, hence λ_3 also remains constant along the curves. Therefore these curves are parallel straight lines. Take any of these straight lines which passes through interior point of Ω_e. Then it can not reach the boundary of Ω_e and must remain in Ω_e. This follows from the fact that for interior point $\lambda_1 \neq \lambda_2$ and for the boundary $\lambda_1 = \lambda_2$. Along characteristic $r_i, i = 1,2,3$ remain constant then also λ_i because they are defined by coefficients a_i which are parameterized by r_1, r_2, r_3. So we have that Ω_e is a strip of the slope $\lambda_3 = const$. Moreover as explained each a_i has constant values along the characteristic. In particular $g = g(x - \lambda_3 t)$. This proves the theorem. □

Now we are in position to complete the proof of main theorem for $n = 3$.

Proof. (Theorem 1). By Theorem 4 Ω_e in coordinate (t,x) is a strip with the slope λ, and $a_i = a_i(x - \lambda t)$, where a_i are functions of one variable. The coefficients a_i satisfy the quasi-linear system
(2)

$$U_t + A(U)U_x = 0, U = (a_0, a_1, a_2)^T,$$

which for $n = 3$ takes the form

$$A(U) = \begin{pmatrix} 0 & 0 & a_1 \\ a_2 & 0 & 2a_2 - 3a_0 \\ 0 & a_2 & 3 - 2a_1 \end{pmatrix}.$$

Since $U = U(x - \lambda t)$ is in the form of the simple wave then U' is λ-eigenvector of $A(U)$. Then one has

$$\begin{aligned} &a_1 a_2' = \lambda a_0' \\ &a_2 a_0' + 2a_2 a_2' - 3a_0 a_2' = \lambda a_1' \\ &a_2 a_1' + (3 - 2a_1)a_2' = \lambda a_2'. \end{aligned} \tag{4}$$

These differential equations can be solved as follows:
Divide the last equation of (4) by a_2^3 to have

$$\frac{a_1}{a_2^2} = \frac{3-\lambda}{2}\frac{1}{a_2^2} + c_1,$$

and so

$$a_1 = (3 - \lambda)/2 + c_1 a_2^2. \tag{5}$$

Divide the second equation of the system (4) by a_2^4 to have

$$(a_0/a_2^3)' = (1/a_2^2)' + \lambda a_1'/a_2^4$$

This expression together with (5) yields

$$\frac{a_0}{a_2^3} = \frac{1-\lambda c_1}{a_2^2} + c_2,$$

which means that

$$a_0 = a_2(1-\lambda c_1) + c_2 a_2^3. \tag{6}$$

On the other hand substituting a_1 from (5) into the first equation of the system (4) one gets

$$\lambda a_0 = \frac{c_1}{3}a_2^3 + \frac{3-\lambda}{2}a_2 + c_3. \tag{7}$$

Eliminating a_0 from the equations (6), (7) one gets certain third power polynomial on a_2 which vanishes. Then there are two possibilities: either the function a_2 is a constant and then the metric is flat (remember $a_2 = g$) or coefficients of this polynomial must vanish. But this yields the identities:

$$c_3 = 0,\ c_1 = \frac{3(\lambda-1)}{2\lambda^2},\ c_2 = \frac{\lambda-1}{3\lambda^3}.$$

Using them one can easily verify the following explicit identity

$$F_3 = k_1 F_1^3 + 2k_2 H F_1,$$

$$k_1 = c_2 = \frac{\lambda-1}{3\lambda^3},\ k_2 = \frac{3-\lambda}{2\lambda}$$

(where $F_3 = \frac{a_0}{a_2^3}p_1^3 + \frac{a_1}{a_2^2}p_1^2 p_2 + p_1 p_2^2 + p_2^3$).

Let us remark here that the case $\lambda = 0$ means $a_i = a_i(x)$ in particular $g = g(x)$ but then $\rho = g^2(x)dt^2 + dx^2$ is obviously flat metric. □

Proof. (Corollary 1). Let us remark first that the fact that the cubic integral can be explicitly expressed through the first power integral is absolutely necessary for the proof. This is because the elliptic domain could be proper subset of the torus. In such a case it is not clear why its coefficients should be constants. Therefore we proceed indirectly as follows.

By the previous theorem we have that the metric ρ posses linear in momenta integral F_1 on the set Ω_e and moreover F_3 can be expressed through F_1 and H. Write $F_1 = b_0(x,y)p_1 + b_1(x,y)p_2$. Using the identity $F_3 = k_1 F_1^3 + 2k_2 H F_1$ and Kolokoltsov constants for F_3 we get b_0 and b_1 must be constants (obviously at least one of them is not zero). But then the last equation of quasi-linear system of the conformal model gives:

$$b_0\left(\frac{1}{\Lambda}\right)_{q_1} + b_1\left(\frac{1}{\Lambda}\right)_{q_2} = 0,$$

therefore $\Lambda = \Lambda(b_1 q_1 - b_0 q_2)$. If we know that Λ is of this form on an open subset of $\mathbb{T}^2$ and if Λ is analytic then obviously Λ is of this form on the whole torus $\mathbb{T}^2$. This completes the proof.

□

5. The case $\Omega_e \neq \mathbb{T}^2$

We shall split our proof in the two cases:

The first is the case when either Ω_e has a nontrivial boundary or the Riemann invariants $r_{1,2}$ becomes real at some point on the torus. And the second case (in the next section), when Ω_e coincides with the whole torus and $r_{1,2}$ are not reals everywhere on it.

Theorem 5. *Let $F_4 : T^*\mathbb{T}^2 \to \mathbb{R}$ be a polynomial of degree 4 such that $\{F_4, H\} = 0$. Denote by Ω_e the domain on $\mathbb{T}^2$ where polynomial $\hat{G}_4$ has two complex-conjugate and two real distinct roots. Then if $\Omega_e \neq \mathbb{T}^2$ or $\Omega_e = \mathbb{T}^2$ but one knows that $\mathrm{Im} r_{1,2} \equiv 0$ then F_4 can be expressed on Ω_e as follows*

$$F_4 = k_1 F_2^2 + 2k_2 H F_2 + 4k_3 H^2$$

where F_2 is a polynomial of degree 2 which is an integral of the geodesic flow.

Proof. Denote by $s_{1,2} = \alpha \pm i\beta$ the pair of complex-conjugate roots of $\hat{G}_4$, and by $r_{1,2} = u \pm iv$ the corresponding Riemann invariants. It follows from the Theorem 3 that $v \equiv 0$ and u is a constant on Ω_e. So we have that $r_{1,2}$ is a real constant. Denote it by r. Then we claim that $F_4 - 4rH^2$ can be factorized $F_4 - 4rH^2 = KM$ where K and M are real polynomials of degree 2 and

$$K = \left(p_2 - \frac{p_1}{g}(\alpha + i\beta)\right)\left(p_2 - \frac{p_1}{g}(\alpha - i\beta)\right).$$

Since $(\alpha \pm i\beta)$ are roots of $\hat{G}_4$ then

$$G_4 = L_v F = (L_v K)M + K(L_v M)$$

must be divisible by K. Therefore there are two possibilities:

1. M and K are relatively prime. In this case $L_v K$ must be divisible by K. Notice that for any quadratic polynomial K, $L_v K$ always have real roots. This is because any function on the circle must have minimum and maximum. Thus the only possibility in the first case is $L_v K \equiv 0$. But this means K is proportional to H. One may assume $K = H$ (by taking the coefficient of proportionality to M). So we have got $F_4 - 4rH^2 = HM$, therefore M must be integral of the geodesic flow degree 2.

2. In this case M is divisible by K, i.e. proportional to K, $M = cK$ (where $c(t, x)$ is a function). So we have $F_4 - 4rH^2 = cK^2$. One has equating the coefficients the coefficients of (p_2^4) at both sides: $1-r = c$. So c is a constant and we have

$$F_4 = 4rH^2 + (1-r)K^2.$$

Again K must be an integral of degree 2. So in both cases we proved that F_4 is reducible. This proves Theorem 5. $\square$

By the same method we can prove the following

Corollary 4. *Under the conditions of Theorem 5 the conformal factor $\Lambda(q_1, q_2)$ can be written on Ω_e in the form*

$$\Lambda(q_1,q_2) = f_1(m_1q_1+n_1q_2)+f_2(m_2q_1+n_2q_2) \quad \textit{with} \quad m_1m_2/n_1n_2 = -1.$$

Moreover, if Λ is known to be real analytic then Λ can be written in such a form for all (q_1, q_2) on $\mathbb{T}^2$.

Proof. We bypass the difficulty exactly as in the previous corollary. As we proved one knows the existence of integral F_2 on the domain Ω_e, such that F_4 can be expressed as function of H and F_2, then write F_2 in conformal model (c) in the complex form $F_2 = b_0p^2 + b_1p\bar{p} + b_2\bar{p}^2$. It follows from Theorem 5 and Kolokoltsov identities that b_0, b_2 must be constants on Ω_e, $b_0 = A + iB, b_2 = A - iB$ and b_1 is a real function. Then the equations on b_1 and Λ are the following:

$$(b_1\Lambda)_{q_1} = -2A\Lambda_{q_1} - 2B\Lambda_{q_2},\ (b_1\Lambda)_{q_2} = -2B\Lambda_{q_1} + 2A\Lambda_{q_2}.$$

Eliminating $(b_1\Lambda)$ we get

$$B\Lambda_{q_1q_1} - 2A\Lambda_{q_1q_2} - B\Lambda_{q_2q_2} = 0$$

which imply the result. $\square$

6. The case $\Omega_e = \mathbb{T}^2$ and $r_{1,2}$ are not real

This is the most difficult case which is left. In this case we shall prove that each one of the real eigenvalues is genuinely nonlinear in the sense of Lax. This fact, together with the property of our system to be Rich (or Semi-Hamiltonian), will enable us to establish blow-up of the derivative r_x unless it vanishes. Such a proof was first given in [3] for other system.

Theorem 6. *Assume $\Omega_e = \mathbb{T}^2$, and assume that for all (t, x) the polynomial $\hat{G}_4$ has 4 distinct roots, 2-complex conjugate $s_{1,2} = \alpha \pm i\beta$ and 2 real $s_{3,4}$. Assume also that the imaginary part of Riemann invariants $r_{1,2}$ does not vanish. Then the real eigenvalues $\lambda_{3,4} = gs_{3,4}$ are necessarily genuinely non-linear and therefore the corresponding Riemann invariants are constants. In particular all a_i must be constant, and so the metric is flat.*

We proceed as follows. Let us first subtract from $F_4 - 4H^2$ in order to make the coefficient of p_2^4 vanish, all the other a_i we shall denote by the same letters.

Lemma 2. *Let $\lambda_3 = gs_3$ be an eigenvalue of our quasi-linear system then the derivative $\frac{\partial \lambda_3}{\partial r_3}$ can be computed*

$$\frac{\partial \lambda_3}{\partial r_3} = -\frac{(1+s_3^2)^2}{S\hat{G}_4'(s_3)}((a_3+a_1)s_3^2 + 4a_0 s_3 - (a_1+a_3))$$

where

$$S = (s_3-s_1)(s_3-s_2)(s_3-s_4),$$

$$\hat{G}_4(s) = a_3 s^4 + 2a_2 s^3 + 3(a_1-a_3)s^2 + (4a_0 - 2a_2)s - a_1.$$

Proof. We have

$$\frac{\partial \lambda_3}{\partial r_3} = \frac{\partial (s_3 a_3)}{\partial r_3} = s_3 \frac{\partial a_3}{\partial r_3} + a_3 \left(\sum_{i=0}^{3} \frac{\partial s_3}{\partial a_i} \frac{\partial a_i}{\partial r_3} \right).$$

Since s_3 is a root of $\hat{G}_4 = 0$ then

$$\frac{\partial s_3}{\partial a_0} = -\frac{4s_3}{\hat{G}_4'(s_3)}, \quad \frac{\partial s_3}{\partial a_1} = \frac{1-3s_3^2}{\hat{G}_4'(s_3)},$$

$$\frac{\partial s_3}{\partial a_2} = \frac{2s_3(1-s_3^2)}{\hat{G}_4'(s_3)}, \quad \frac{\partial s_3}{\partial a_3} = \frac{s_3^2(s_3^2-3)}{\hat{G}_4'(s_3)}.$$

In order to find $\frac{\partial \lambda_3}{\partial r_3}$ we need to calculate $\frac{\partial a_i}{\partial r_3}$:

$$\left(\frac{\partial a_i}{\partial r_j} \right) = \left(\frac{\partial F(s_i)}{\partial a_j} \right)^{-1}.$$

From the previous identity we obtain

$$\frac{\partial a_0}{\partial r_3} = -\frac{(1+s_3^2)^2 s_1 s_2 s_4}{S}, \quad \frac{\partial a_1}{\partial r_3} = \frac{(1+s_3^2)^2 (s_1 s_2 + s_1 s_4 + s_2 s_4)}{S},$$

$$\frac{\partial a_2}{\partial r_3} = -\frac{(1+s_3^2)^2 (s_1 + s_2 + s_4)}{S}, \quad \frac{\partial a_3}{\partial r_3} = \frac{(1+s_3^2)^2}{S}.$$

The fact that $s_i, i = 1, \ldots, 4$ are roots of $\hat{G}_4$ gives us

$$s_1 + s_2 + s_4 = -\frac{2a_2}{a_3} - s_3,$$

$$s_1 s_2 + s_1 s_4 + s_2 s_4 = \frac{3(a_1+a_3)}{a_3} - s_3(a_1 + s_2 + s_4),$$

$$\frac{3(a_1+a_3)}{a_3} + s_3 \left(\frac{2a_2}{a_3} + s_3 \right),$$

$$s_1 s_2 s_4 = -\frac{a_1}{a_3 s_3},$$

and

$$\frac{\partial a_0}{\partial r_3} = \frac{(1+s_3^2)^2 a_1}{a_3 s_3 S}, \quad \frac{\partial a_1}{\partial r_3} = \frac{(1+s_3^2)^2 (3(a_1-a_3) + 2a_2 s_3 + a_3 s_3^2)}{a_3 S},$$

$$\frac{\partial a_2}{\partial r_3} = \frac{(1+s_3^2)^2 (2a_2 + a_3 s_3)}{a_3 S}.$$

From here we have

$$\frac{\partial \lambda_3}{\partial r_3} = -\frac{(1+s_3^2)^2}{S\hat{G}_4'(s_3)}(2a_3s_3^4+4a_2s_3^3+3(a_1-3a_3)s_3^2-4(a_0+a_2)s_3+a_1+3a_3).$$

The identity

$$2\hat{G}_4(s_3) - (2a_3s_3^4 + 4a_2s_3^3 + 3(a_1-3a_3)s_3^2 - 4(a_0+a_2)s_3 + a_1 + 3a_3)$$
$$(a_3+a_1)s_3^2 + 4a_0s_3 - (a_1+a_3).$$

proves the Lemma 2. □

Genuine nonlinearity is proved in the following lemma which is in fact general fact about critical values of the polynomials on the circle.

Lemma 3. *Let F_4 be a polynomial of degree 4 $F_4 = \sum_{i=1}^{3} a_i \left(\frac{p_1}{g}\right)^{n-i} p_2^i$ considered on the circle $H = \frac{1}{2}\left(\left(\frac{p_1}{g}\right)^2 + p_2^2\right) = \frac{1}{2}$. Denote by G_4 the derivative of F_4. Let s_i be the roots of $\hat{G}_4$ such that $s_{1,2}$ are complex conjugate and $s_{3,4}$ are real distinct. Let r_i denote critical values, with $r_{1,2}$ are complex conjugate (not real) and $r_{3,4}$ real. Then it follows that for $\lambda_i = gs_i$*

$$\frac{\partial \lambda_3}{\partial r_3} \neq 0 \quad \textit{and} \quad \frac{\partial \lambda_4}{\partial r_4} \neq 0.$$

Proof. In order to prove the lemma denote

$$s_{1,2} = \alpha \pm i\beta,\ \beta \neq 0.$$

We have to show that in such a case polynomials $\hat{G}_4(s)$ and

$$\gamma = (a_3+a_1)s^2 + 4a_0s - (a_1+a_3)$$

are relatively prime (notice that γ has only real roots). This can be done as follows. Divide $\hat{G}_4$ by γ with a reminder R. If $\hat{G}_4$ and γ have common root then R has to be of degree zero or to be equal to zero. Dividing explicitly one gets for R the following expression:

$$R$$
$$\frac{2(a_1^3 + a_1^2a_3 - a_1(4a_0a_2 + a_3^2) + a_3(8a_0^2 - 4a_0a_2 - a_3^2))(a_1 + a_3 - 4a_0s)}{(a_1+a_3)^3}.$$

Notice that R can be a number in two cases only:

Case 1. $R \neq 0$ but $a_0 = 0$.

In this case γ has two roots ± 1 and the value of $\hat{G}_4$ is the same for both of them

$$\hat{G}_4(\pm 1) = 2(a_1 - a_3).$$

But this means that if one of them is a common root of $\hat{G}_4$ and γ then in fact both of them are. But in the first case R is not zero, contradiction.

Case 2. $R = 0$.

In this case $\hat{G}_4$ is divisible by γ. Denote $\alpha \pm i\beta, \mu, -1/\mu$ the roots of $\hat{G}_4$ where the last two are the roots of γ. With these notations one obtain from the Viete's formula the relation between them

$$\alpha^2 + \beta^2 - 1 = \alpha(\mu - 1/\mu).$$

Notice that if $\alpha = 0$ then $\beta = 1$ and this case is excluded by Lemma 1. Moreover the Viete's formulas together with this relation lead to the following expressions

$$\frac{a_0}{a_3} = \frac{1 - (\alpha^2 + \beta^2)^2}{4\alpha},$$

$$\frac{a_1}{a_3} = \alpha^2 + \beta^2,$$

$$\frac{a_2}{a_3} = \frac{1 - 3\alpha^2 - \beta^2}{2\alpha}.$$

Using these formulas one can substitute them into the value of Riemann invariant of the root $\alpha + i\beta$ to get by direct calculations that its imaginary part vanishes identically:

$$v = \mathrm{Im}\frac{\hat{F}_4(\alpha + i\beta)}{(1 + (\alpha + i\beta)^2)^2} =$$

$$\mathrm{Im}\frac{a_0 + a_1(\alpha + i\beta) + a_2(\alpha + i\beta)^2 + a_3(\alpha + i\beta)^3}{(1 + (\alpha + i\beta)^2)^2} = 0.$$

But this is not possible by the assumptions.

The exceptional cases can be treated analogously:

Case 3. It could happen that γ is identically zero, i.e.

$$a_1 + a_3 = 0, \ a_0 = 0.$$

In this case $\frac{\hat{G}_4}{a_3} = s^4 + 2ks^3 - 6s^2 - 2ks - 1$, where $k = \frac{a_2}{a_3}$. One can check that this is impossible because such a polynomial has 4 real roots. One can see this for instance through $\frac{\hat{F}_4}{(1+s^2)^2}$ which in this case is

$$\frac{\hat{F}_4}{(1 + s^2)^2} = \frac{-a_3 s + a_2 s^2 + s_3 s^3}{(1 + s^2)^2}.$$

But the expression in the nominator has zero as a root and two more real roots. Therefore $\hat{G}_4$ must have 4 real roots. Contradiction.

Case 4. In this case it could happen that $a_1 + a_3 = 0$ and the common root of $\hat{G}_4$ and γ is $s = 0$. But then $a_1 = 0$. Then also $a_3 = 0$ but this is not possible since $a_3 = g$ is always positive. This finishes the proof of lemma.

□

Proof. (Theorem 6). It follows from Lemma 2 that real eigenvalues are genuinely non-linear. It was shown in our previous paper [4] that our system is rich (Semi–Hamiltonian). This property is crucial because it enables to use Lax analysis of blow up along characteristics as it was done for Benney chain in [3]. This method uses Ricatti type equation which $(r_3)_x$ and $(r_4)_x$ must satisfy, and if one knows that λ_3, λ_4 are genuinely non-linear (Lemma 2) then $(r_3)_x = (r_4)_x = 0$ since otherwise there is a blow up after a finite time for Ricatti equation. We had before already $r_1 = const, r_2 = const$. Thus all Riemann invariants are constant and so a_i. In particular g is constant and the Riemann metric is flat. □

7. Concluding remarks and questions

1. It would be very interesting to know if our Semi-Hamiltonian system (2) is in fact Hamiltonian and to find Dubrovin–Novikov bracket of hydrodynamic type (see [7]).

2. In this paper we show that in the case $n = 3, 4$ the system (2) standard in the elliptic domain Ω_e. It follows from our main theorems that in the analytic case we can assume that for $n = 3, 4$ hyperbolic domain is whole torus $\mathbb{T}^2$. It follows from [17] that in Hyperbolic domain Semi-Hamiltonian systems can be integrated by Tsarev's generalized hodograph method. It would be very interesting to apply this method to the system (2).

3. It would be natural to find the blow up mechanism for the quasilinear system in the conformal model (c). Technically this is not obvious because the characteristic fields may rotate on the torus.

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